

Pontryagin's Minimum Principle. Optimal
Control with Constraints on Inputs

Examples of optimal control problems. Variational approach

1.1 The optimal control problem, a summary of necessary conditions and Pontryagin minimum principle

The problem is to find an admissible control $\mathbf{u}^*(t)$ that causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (1.1)$$

to follow an admissible trajectory $\mathbf{x}^*(t)$ that minimizes the performance measure:

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) dt \quad (1.2)$$

We shall initially assume that the admissible state and control are not bounded and that the initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ and the initial time t_0 are specified. $\mathbf{x}(t)$ is the $n \times 1$ state vector and $\mathbf{u}(t)$ is the $m \times 1$ vector of control inputs.

The *Hamiltonian* is defined as:

$$H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) = g(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda^T(t) \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) = g + \lambda^T \mathbf{f}$$

where $\lambda(t)$ is a vector of n unknown functions of time, also called the *co-states*.

Regardless of any constraints on the control vector $\mathbf{u}(t)$, Lev Pontryagin had shown that $\mathbf{u}(t)$ must be chosen to minimize the Hamiltonian. A rigorous proof of the fact that $\mathbf{u}(t)$ must be chosen to optimize the Hamiltonian is

Pontryagin's most notable contribution to optimal control theory. For this reason, the approach is often called Pontryagin Principle, (Naidu, 2002).

The necessary conditions for optimal control, as resulted from the previous chapter can be used to find candidate solutions of the problem as follows:

1. Write the Hamiltonian:

$$H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) = g(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda^T(t) \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) = g + \lambda^T \mathbf{f}$$

then write the necessary conditions:

2. The first derivative of the hamiltonian with respect to $\mathbf{u}$:

$$\frac{\partial H}{\partial \mathbf{u}}(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) = 0 \quad (1.3)$$

According to the Pontryagin's principle, $\mathbf{u}(t)$ must minimize the Hamiltonian, thus, the second derivative must be computed and compared to 0.

3. The costate equation is

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) \quad (1.4)$$

4. The state equation is:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad (1.5)$$

Equations (1.4) and (1.5) are a set of $2n$ differential equations with $2n$ unknown variables ($\mathbf{x}(t)_{n \times 1}$ and $\lambda(t)_{n \times 1}$). A unique solution requires a set of $2n$ boundary conditions. They can be stated or computed as in the following cases:

Case I. The final time t_f is specified, the final state $\mathbf{x}(t_f)$ is specified.

$$\mathbf{x}(t_f) = \mathbf{x}_f \quad (1.6)$$

are n relations which are boundary conditions for differential equations. Together with the initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ will ensure a unique solution of the system of differential equations (1.4) and (1.5).

Case II. The final time t_f is specified, the final state is free. The final condition can be calculated from the first transversality condition:

$$\lambda(t_f) = \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t_f), t_f) \quad (1.7)$$

Case III. The final time t_f is free, the final state $\mathbf{x}(t_f) = \mathbf{x}_f$ is fixed. From the second transversality condition (see Calculus of variations) we obtain:

$$H(\mathbf{x}(t_f), \mathbf{u}(t_f), \lambda(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}(t_f), t_f) = 0 \quad (1.8)$$

Case IV. The final time t_f is free, the final state is free. The two boundary conditions are derived from the two transversality conditions:

$$\lambda(t_f) = \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t_f), t_f) \quad (1.9)$$

$$H(\mathbf{x}(t_f), \mathbf{u}(t_f), \lambda(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}(t_f), t_f) = 0 \quad (1.10)$$

1.2 Examples

1.2.1 A minimum energy problem

Consider the problem of accelerating a skateboard in such a way as to maximize the total distance traveled in a given time T , minus the effort expended, (Weber, 2000). Denote $x_1(t)$ the distance traveled at time t . The speed $x_2(t)$ is the first derivative of x_1 , and let the acceleration be $u(t)$, the first derivative of x_2 . The dynamical system that describes the problem is:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= u(t) \end{aligned} \quad (1.11)$$

where $x_1(0) = 0$, $x_2(0) = 0$. The performance measure to be minimized in this case is:

$$J = -x_1(T) + \frac{1}{2} \int_0^T u^2(t) dt \quad (1.12)$$

Note that the first term in J is the negative distance traveled in time T , i.e. while this term will be minimized, the distance $x_1(T)$ is maximized. The squared acceleration integrated over the entire time interval corresponds to the effort expended in time T .

The hamiltonian is

$$H(\mathbf{x}, u, \lambda) = \frac{1}{2} u^2(t) + \lambda_1(t) x_2(t) + \lambda_2(t) u(t)$$

The control u^* that minimizes the hamiltonian is computed from

$$\frac{\partial H}{\partial u} = u(t) + \lambda_2(t) = 0$$

or

$$u^*(t) = -\lambda_2(t)$$

This control will minimize indeed the hamiltonian since the second derivative is positive, or

$$\frac{\partial^2 H}{\partial u^2} = 1 > 0$$

The costates are computed from:

$$\begin{aligned}\dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x_1} = 0 \\ \dot{\lambda}_2(t) &= -\frac{\partial H}{\partial x_2} = -\lambda_1(t)\end{aligned}\tag{1.13}$$

Since the final states are free and the final time is given (T), the final conditions for $\lambda_{1,2}$ are given by:

$$\begin{aligned}\lambda_1(T) &= \frac{\partial h(\mathbf{x}(T))}{\partial x_1(T)} = -1 \\ \lambda_2(T) &= \frac{\partial h(\mathbf{x}(T))}{\partial x_2(T)} = 0\end{aligned}\tag{1.14}$$

where

$$h(\mathbf{x}(T)) = -x_1(T)$$

From (1.13) and (1.14) we obtain:

$$\begin{aligned}\lambda_1(t) &= -1 \\ \lambda_2(t) &= t - T\end{aligned}\tag{1.15}$$

Because $u^*(t) = -\lambda_2(t)$, we obtain the optimal control law as:

$$u^*(t) = T - t$$

which decreases linearly with time and reaches zero at the final time.

1.2.2 Zermelo's problem

The formulation of Zermelo's problem has been addressed in a previous chapter.

Consider a ship traveling from point A to point B in rectangular cartesian coordinates (x, y) , through a region of strong currents as shown in Figure 1.1, (Longuski et al., 2013; Weber, 2000). The ship is assumed to have a constant speed V with respect to the water, but its heading angle θ can be varied. The

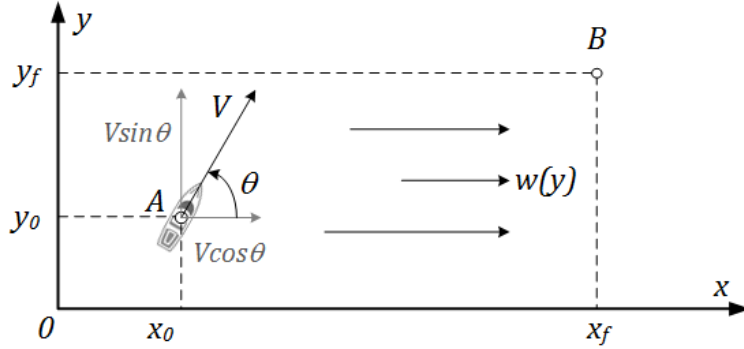


Figure 1.1: Zermelo's problem

velocity of the water current, $w(y)$, is parallel to the x -axis and varies with y . The function w may be a constant value, or a function of time and/or the state variable $y(t)$.

The goal is to steer the boat from the initial position $A(x_0, y_0)$ to a given point $B(x_f, y_f)$ in minimum time.

The optimal control problem can be stated as follows: find the optimal control $\theta^*(t)$ that minimizes the performance measure:

$$J = \int_0^{t_f} dt$$

subject to:

$$\begin{aligned}\dot{x}(t) &= V \cos \theta(t) + w(y(t)) \\ \dot{y}(t) &= V \sin \theta(t)\end{aligned}$$

where $t \in [0, t_f]$ and $x(0) = x_0$, $y(0) = y_0$, $x(t_f) = x_f$, $y(t_f) = y_f$.

Solution of the Zermelo's problem

Simple Zermelo problem. Consider the simple case when the water velocity is zero $w(y) = 0$. The coordinates of the initial and final points are $A(0, 0)$ and $B(1, 1)$.

The system dynamics is given by:

$$\begin{aligned}\dot{x}(t) &= V \cos \theta(t) \\ \dot{y}(t) &= V \sin \theta(t)\end{aligned}\tag{1.16}$$

where $t \in [0, t_f]$ and the boundary conditions (constraints) for the states are: $x(0) = 0$, $y(0) = 0$, $x(t_f) = 1$, $y(t_f) = 1$.

The goal is to determine the unconstrained steering angle $\theta^*(t)$ that minimizes the cost function:

$$J = \int_0^{t_f} dt$$

The expected optimal trajectory between A and B for no current is a straight line between points $(0, 0)$ and $(1, 1)$, that means a constant angle $\theta^* = 45^\circ$ has to be applied. This will be demonstrated in the followings using Pontryagin's minimum principle.

- The Hamiltonian for the system is:

$$H = 1 + \lambda_1 V \cos \theta(t) + \lambda_2 V \sin \theta(t)$$

- The control input is unconstrained so we have:

$$\frac{\partial H}{\partial \theta} = -\lambda_1 V \sin \theta(t) + \lambda_2 V \cos \theta(t) = 0$$

or

$$\lambda_1 V \sin \theta(t) = \lambda_2 V \cos \theta(t)$$

and

$$\frac{\sin \theta(t)}{\cos \theta(t)} = \tan \theta(t) = \frac{\lambda_2(t)}{\lambda_1(t)} \quad (1.17)$$

- The co-states must satisfy the equations:

$$\dot{\lambda}_1(t) = -\frac{\partial H}{\partial x} = 0$$

$$\dot{\lambda}_2(t) = -\frac{\partial H}{\partial y} = 0$$

because the hamiltonian does not depend on the states x and y . Thus, λ_1 and λ_2 are constants:

$$\lambda_1 = C_1, \quad \lambda_2 = C_2$$

and from (1.17) we obtain that:

$$\tan \theta(t) = \frac{C_2}{C_1} = \text{constant}, \quad \Rightarrow \quad \theta(t) = \arctan \left(\frac{C_2}{C_1} \right) = \theta^* = \text{constant}$$

- Solving the state equations (1.16) with both V and θ constant gives:

$$\begin{aligned}x(t) &= (V \cos \theta^*)t + k_1 \\y(t) &= (V \sin \theta^*)t + k_2\end{aligned}$$

Evaluating this solution at $t = 0$ will give the values of the integration constants k_1 and k_2 :

$$\begin{aligned}x(0) &= (V \cos \theta^*) \cdot 0 + k_1 = 0 \Rightarrow k_1 = 0 \\y(0) &= (V \sin \theta^*) \cdot 0 + k_2 = 0 \Rightarrow k_2 = 0\end{aligned}$$

- Evaluating at $t = t_f$:

$$\begin{aligned}x(t_f) &= (V \cos \theta^*) \cdot t_f = 1 \\y(t_f) &= (V \sin \theta^*) \cdot t_f = 1\end{aligned}$$

so

$$\tan \theta^* = 1 \Rightarrow \theta^* = 45^\circ$$

Another particular case. The problem can be reformulated in a more general case when the water stream has a constant velocity w in the direction of x -axis. The system dynamics in this case is

$$\begin{aligned}\dot{x}(t) &= V \cos \theta(t) + w \\ \dot{y}(t) &= V \sin \theta(t)\end{aligned} \tag{1.18}$$

where $t \in [0, t_f]$ and $x(0) = 0$, $y(0) = 0$, $x(t_f) = 0$, $y(t_f) = 1$. Note that the final position is at a distance of 1 unit in the direction of y -axis.

The objective is to find the steering angle $\theta^*(t)$ that minimizes the final time:

$$J = \int_0^{t_f} dt$$

- The Hamiltonian for the system is:

$$H = 1 + \lambda_1(V \cos \theta(t) + w) + \lambda_2 V \sin \theta(t)$$

- The control input is unconstrained so we have:

$$\frac{\partial H}{\partial \theta} = -\lambda_1 V \sin \theta(t) + \lambda_2 V \cos \theta(t) = 0$$

or

$$\lambda_1 V \sin \theta(t) = \lambda_2 V \cos \theta(t)$$

and

$$\frac{\sin \theta(t)}{\cos \theta(t)} = \tan \theta(t) = \frac{\lambda_2(t)}{\lambda_1(t)} \tag{1.19}$$

- The co-states must satisfy the equations:

$$\dot{\lambda}_1(t) = -\frac{\partial H}{\partial x} = 0$$

$$\dot{\lambda}_2(t) = -\frac{\partial H}{\partial y} = 0$$

because the hamiltonian does not depend on the states x and y .
Thus, λ_1 and λ_2 are constants:

$$\lambda_1 = C_1, \quad \lambda_2 = C_2$$

and from (1.19) we obtain that:

$$\tan \theta(t) = \frac{C_2}{C_1} = \text{constant}, \quad \Rightarrow \quad \theta(t) = \arctan\left(\frac{C_2}{C_1}\right) = \theta^* = \text{constant}$$

- Solving the state equations (1.16) with both V and θ constant gives:

$$\begin{aligned} x(t) &= (V \cos \theta^* + w)t + k_1 \\ y(t) &= (V \sin \theta^*)t + k_2 \end{aligned}$$

Evaluating this solution at $t = 0$ will give the values of the integration constants k_1 and k_2 :

$$\begin{aligned} x(0) &= (V \cos \theta^* + w) \cdot 0 + k_1 = 0 \quad \Rightarrow \quad k_1 = 0 \\ y(0) &= (V \sin \theta^*) \cdot 0 + k_2 = 0 \quad \Rightarrow \quad k_2 = 0 \end{aligned}$$

- Evaluating at $t = t_f$:

$$\begin{aligned} x(t_f) &= (V \cos \theta^* + w) \cdot t_f = 0 \\ y(t_f) &= (V \sin \theta^*) \cdot t_f = 1 \end{aligned} \tag{1.20}$$

so

$$\cos \theta^* = -\frac{w}{V}, \quad \text{and} \quad \sin \theta^* = \frac{1}{Vt_f}$$

Rearranging we get:

$$\sin \theta^* = \sqrt{1 - \cos^2 \theta^*} = \sqrt{1 - \frac{w^2}{V^2}} = \frac{\sqrt{V^2 - w^2}}{V} = \frac{1}{Vt_f}$$

then:

$$t_f = \frac{1}{\sqrt{V^2 - w^2}}, \quad \text{and} \quad \theta^* = \arccos\left(-\frac{w}{V}\right) = \pi - \arccos\left(\frac{w}{V}\right)$$

or

$$\theta^* = \arcsin \frac{\sqrt{V^2 - w^2}}{V}$$

Note that the final time t_f is real positive only if $V > w$ (the boat velocity is greater than the water stream velocity).

A new performance measure. Let us reformulate the previous problem as follows: find the steering angle $\theta^*(t)$ to maximize the travel distance in the x -direction, in a given time t_f . Assume that the initial position of the boat is $x(0) = y(0) = 0$ and the water stream velocity is linear with respect to y : $w(y(t)) = y(t)$.

The system dynamics is:

$$\begin{aligned}\dot{x}(t) &= V \cos \theta(t) + y(t) \\ \dot{y}(t) &= V \sin \theta(t)\end{aligned}\tag{1.21}$$

The initial conditions are $x(0) = 0$, $y(0) = 0$, the final time t_f is fixed (given) and the performance measure to be minimized is:

$$J = -x(t_f)$$

Note that $x(t_f)$ in J has a negative sign because the objective is to maximize $x(t_f)$, i.e. minimize $-x(t_f)$.

- The Hamiltonian for the system is:

$$H = 0 + \lambda_1(V \cos \theta(t) + y(t)) + \lambda_2 V \sin \theta(t)$$

- The control input is unconstrained so we have:

$$\frac{\partial H}{\partial \theta} = -\lambda_1 V \sin \theta(t) + \lambda_2 V \cos \theta(t) = 0$$

or

$$\lambda_1 V \sin \theta(t) = \lambda_2 V \cos \theta(t)$$

and

$$\frac{\sin \theta(t)}{\cos \theta(t)} = \tan \theta(t) = \frac{\lambda_2(t)}{\lambda_1(t)}\tag{1.22}$$

- The co-states must satisfy the equations:

$$\dot{\lambda}_1(t) = -\frac{\partial H}{\partial x} = 0$$

$$\dot{\lambda}_2(t) = -\frac{\partial H}{\partial y} = -\lambda_1$$

because the hamiltonian does not depend on the state x but it depends on y . Thus:

$$\lambda_1 = C_1, \quad \lambda_2 = -C_1 t + C_2$$

- The final conditions for the states are not imposed, therefore, the final conditions for the co-states are:

$$\begin{aligned} \lambda_1(t_f) &= \frac{\partial(-x(t_f))}{\partial x(t_f)} = -1 \\ \lambda_2(t_f) &= \frac{\partial(0)}{\partial x(t_f)} = 0 \end{aligned}$$

and then we have:

$$\lambda_1(t_f) = -1 = C_1, \quad \lambda_2(t_f) = -(-1)t_f + C_2 = 0 \Rightarrow C_1 = -1, \quad C_2 = -t_f$$

or

$$\lambda_1(t) = -1, \quad \lambda_2(t) = t - t_f$$

- The optimal $\theta(t)$ is obtained from (1.22):

$$\tan \theta^*(t) = \frac{\lambda_2(t)}{\lambda_1(t)} = \frac{t - t_f}{-1} = t_f - t, \Rightarrow \theta^*(t) = \arctan(t_f - t)$$

Pontryagin's Minimum Principle

2.1 Constrained optimal control and the minimum principle

We have assumed that the admissible controls and states are not constrained by any boundaries; however, in realistic systems such constraints commonly occur. Physically realizable controls generally have magnitude limitations: the thrust of a rocket engine cannot exceed a certain value; motors, which provide torque, saturate; attitude control mass expulsion systems are capable of providing a limited torque. State constraints often arise because of safety or structural restrictions: the current in an electric motor cannot exceed a certain value without damaging the windings; the turning radius of a maneuvering aircraft cannot be less than a specified minimum value (Kirk, 2004).

The general approach in which we consider the effect of control constraints and show the necessary conditions leads to the *Pontryagin's minimum principle*.

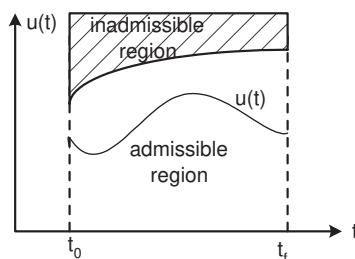


Figure 2.1: An extremal control that is constrained by a boundary

Figure 2.1 shows an example of bounded controls. All functions bounded by vertical lines between t_0 and t_f and placed in the admissible region are *admissible controls*.

It can be demonstrated (Kirk, 2004) that a necessary condition for $u^*(t)$ to *minimize* the functional J is:

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t), t) \leq H(\mathbf{x}^*(t), \mathbf{u}(t), \lambda^*(t), t) \quad (2.1)$$

for all $t \in [t_0, t_f]$ and for all *admissible controls*.

Equation (2.1) which indicates that *an optimal control must minimize the hamiltonian* is called **Pontryagin's minimum principle**.

Notice that we have established a necessary condition but not (in general) sufficient condition for optimality. An optimal control must satisfy Pontryagin's minimum principle; however, there may be controls that satisfy the minimum principle that are not optimal.

Let us now summarize the principal results.

The problem is to find a control $u^*(t) \in U$ (where U is the class of admissible controls), which causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad (2.2)$$

to follow an admissible trajectory that minimizes the performance measure

$$J(\mathbf{u}) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) dt \quad (2.3)$$

In terms of the hamiltonian:

$$H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) = g(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda^T(t) \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad (2.4)$$

the necessary conditions for $\mathbf{u}^*(t)$ to be optimal are:

$$\dot{\mathbf{x}}^*(t) = \frac{\partial H}{\partial \lambda}(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t), t) = \mathbf{f}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) \quad (2.5)$$

$$\dot{\lambda}^*(t) = -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t), t) \quad (2.6)$$

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t), t) \leq H(\mathbf{x}^*(t), \mathbf{u}(t), \lambda^*(t), t) \quad (2.7)$$

for all admissible controls $\mathbf{u}(t)$ and for all $t \in [t_0, t_f]$

The following boundary conditions hold, when the final state is free and the final time is fixed:

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) = \lambda^*(t_f) \quad (2.8)$$

and:

$$H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \lambda^*(t_f), t_f) + \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) = 0 \quad (2.9)$$

when the final state is fixed and the final time is free.

It should be emphasized that:

- $\mathbf{u}^*(t)$ is a control that causes $H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t), t)$ to assume its *global*, or absolute minimum.
- Equations (2.5), (2.6), (2.7), (2.8) constitute a set of necessary conditions for optimality; these conditions are not, in general, sufficient.

In addition, the minimum principle, although derived for controls with values in a closed and bounded region, can also be applied to problems in which the admissible controls are not bounded. This can be done by viewing the unbounded control region as having arbitrarily large bounds, thus ensuring that the optimal control will not be constrained by boundaries. In this case, for $u^*(t)$ to *minimize* the hamiltonian it is necessary (but not sufficient) that

$$\frac{\partial H}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t), t) = 0 \quad (2.10)$$

If equation (2.10) is satisfied, and the matrix:

$$\frac{\partial^2 H}{\partial \mathbf{u}^2}(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t), t)$$

is positive definite, this is sufficient to guarantee that $\mathbf{u}^*(t)$ causes H to be a *local* minimum; if the hamiltonian can be expressed in the form:

$$H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) = \mathbf{f}(\mathbf{x}, \lambda, t) + \mathbf{c}(\mathbf{x}, \lambda, t)^T \mathbf{u}(t) + \frac{1}{2} \mathbf{u}^T \mathbf{R}(t) \mathbf{u} \quad (2.11)$$

where $\mathbf{c}$ is a $m \times 1$ array that does not have any terms containing $\mathbf{u}(t)$, then satisfaction of (2.10) and $\frac{\partial^2 H}{\partial \mathbf{u}^2}$ be positive definite are *necessary and sufficient* for $H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t), t)$ to be a *global minimum*.

For H of the form (2.11)

$$\frac{\partial^2 H}{\partial \mathbf{u}^2}(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t), t) = \mathbf{R}(t)$$

thus, if $\mathbf{R}(t)$ is positive definite

$$\mathbf{u}^*(t) = -\mathbf{R}^{-1}(t) \mathbf{c}(\mathbf{x}^*(t), \lambda^*(t), t)$$

minimizes the hamiltonian.

Example 2.1 *Let us now illustrate the effect on the necessary conditions of constraining the admissible control values. Consider the system having the state equations:*

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) + u(t)\end{aligned}\tag{2.12}$$

with the initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$. The performance measure to be minimized is:

$$J(u) = \int_{t_0}^{t_f} \frac{1}{2} [x_1^2(t) + u^2(t)] dt \tag{2.13}$$

t_f is specified and the final state $\mathbf{x}(t_f)$ is free.

a) *Find the necessary conditions for an unconstrained control to minimize J .*

$$H(\mathbf{x}, u, \lambda) = \frac{1}{2} [x_1^2(t) + u^2(t)] + \lambda_1 x_2(t) + \lambda_2(t) (-x_2(t) + u(t)) \tag{2.14}$$

The costate equations are:

$$\begin{aligned}\dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x_1} = -x_1(t) \\ \dot{\lambda}_2(t) &= -\frac{\partial H}{\partial x_2} = -\lambda_1(t) + \lambda_2(t)\end{aligned}\tag{2.15}$$

Since the control values are unconstrained, it is necessary that:

$$\frac{\partial H}{\partial u} = u^*(t) + \lambda_2(t) = 0 \Rightarrow u^*(t) = -\lambda_2^*(t) \tag{2.16}$$

Notice that the hamiltonian is of the form (2.11) and

$$\frac{\partial^2 H}{\partial u^2} = 1 > 0$$

therefore $u^*(t)$ does minimize the hamiltonian.

The costate boundary conditions are:

$$\begin{aligned}\lambda_1(t_f) &= \frac{\partial h}{\partial x_1}(\mathbf{x}(t_f), t_f) = 0 \\ \lambda_2(t_f) &= \frac{\partial h}{\partial x_2}(\mathbf{x}(t_f), t_f) = 0\end{aligned}\tag{2.17}$$

since h does not appear in J .

b) Find necessary conditions for optimal control if:

$$-1 \leq u(t) \leq 1, \text{ for all } t \in [t_0, t_f] \quad (2.18)$$

The state and costate equations and the boundary conditions for $\lambda(t_f)$ remain unchanged; however u must be selected to minimize

$$H(\mathbf{x}^*, u, \lambda^*) = \frac{1}{2}x_1^{2*} + \frac{1}{2}u^2 + \lambda_1^*x_2^* + \lambda_2^*(-x_2^* + u)$$

subject to the constraining relations (2.18). To determine the control that minimizes H , we first separate all of the terms containing $u(t)$

$$\frac{1}{2}u^2(t) + \lambda_2^*(t)u(t) \quad (2.19)$$

from the hamiltonian. For times when the optimal control is unsaturated, we have:

$$u^*(t) = -\lambda_2^*(t)$$

as in part a); clearly this will occur when $|\lambda_2^*(t)| \leq 1$. If however, there are times when $|\lambda_2^*(t)| > 1$ then, from (2.19) the control that minimizes H is:

$$u^*(t) = \begin{cases} -1 & \text{for } \lambda_2^*(t) > 1 \\ 1 & \text{for } \lambda_2^*(t) < -1 \end{cases}$$

Thus $u^*(t)$ is the saturated function of $\lambda_2^*(t)$ pictured in Figure 2.2. For

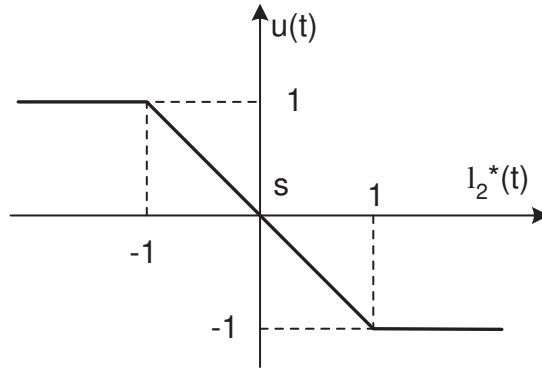


Figure 2.2: Saturated control

the constrained control we have:

$$u^*(t) = \begin{cases} -1 & \text{for } \lambda_2^*(t) > 1 \\ -\lambda_2^*(t) & \text{for } -1 \leq \lambda_2^*(t) \leq 1 \\ 1 & \text{for } \lambda_2^*(t) < -1 \end{cases} \quad (2.20)$$

To determine $u^*(t)$ explicitly, the state and costate equations must be solved. Because of the differences in equations (2.16) and (2.20) the state-costate trajectories in the two cases will be the same only if the initial state values are such that the bounded control does not saturate. If this situation occurs, the control constraints do not affect the solution.

Note that the optimal control history for part b) *cannot* be determined, in general, by calculating the optimal control history for part a) and allowing to saturate whenever the stipulated boundaries are violated.

2.2 Additional necessary conditions

Other necessary conditions may be obtained for the special case when the hamiltonian does not depend explicitly on time, or the optimal control problem is stated as follows:

Determine the optimal control $u^*(t)$ that causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad t_0 \leq t \leq t_f \quad (2.21)$$

to follow an optimal trajectory $\mathbf{x}^*(t)$, while minimizing the performance measure:

$$J = h(\mathbf{x}(t_f)) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t)) dt \quad (2.22)$$

Additional condition 1: If the final time t_f is fixed and the Hamiltonian does not depend explicitly on time, the hamiltonian is constant along the optimal trajectory, or:

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t)) = C, \quad t_0 \leq t \leq t_f \quad (2.23)$$

Additional condition 2: If the final time t_f is free and the Hamiltonian does not depend explicitly on time, the hamiltonian is zero along the optimal trajectory, or:

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t)) = 0, \quad t_0 \leq t \leq t_f \quad (2.24)$$

One approach to prove the relation (2.23) can be to differentiate the hamiltonian with respect to time:

$$\begin{aligned} \frac{d}{dt} H(x(t), u(t), \lambda(t)) &= \frac{d}{dt} \left(g(\mathbf{x}(t), \mathbf{u}(t)) + \lambda^T(t) \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \right) = \\ &= \left[\frac{d\mathbf{x}(t)}{dt} \right]^T \frac{\partial H}{\partial \mathbf{x}} + \left[\frac{d\lambda(t)}{dt} \right]^T \frac{\partial H}{\partial \lambda} + \left[\frac{d\mathbf{u}(t)}{dt} \right]^T \frac{\partial H}{\partial \mathbf{u}} = \\ &= \dot{\mathbf{x}}(t)^T \frac{\partial H}{\partial \mathbf{x}} + \dot{\lambda}(t)^T \frac{\partial H}{\partial \lambda} + \dot{\mathbf{u}}(t)^T \frac{\partial H}{\partial \mathbf{u}} \end{aligned} \quad (2.25)$$

Using the necessary conditions for the optimal trajectory $\mathbf{x}^*(t)$:

$$\dot{\mathbf{x}}^*(t) = \frac{\partial H}{\partial \lambda} \quad (2.26)$$

$$\dot{\lambda}^*(t) = -\frac{\partial H}{\partial \mathbf{x}} \quad (2.27)$$

and knowing that the optimal control $\mathbf{u}^*(t)$ satisfies:

$$\frac{\partial H}{\partial \mathbf{u}} = 0 \quad (2.28)$$

the relation (2.25) becomes:

$$\frac{dH}{dt} = \dot{\mathbf{x}}^*(t)^T(-\dot{\lambda}^*(t)) + \dot{\lambda}^*(t)^T \dot{\mathbf{x}}^*(t) + \dot{\mathbf{u}}^*(t)^T \cdot 0 = 0 \quad (2.29)$$

which means the hamiltonian is constant when evaluated along the optimal trajectory.

The relation (2.24) results directly from the boundary condition written for the case when the final time is free, i.e.:

$$H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \lambda^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0 \quad (2.30)$$

When the hamiltonian and the function h do not depend explicitly on time, the relation (2.30) becomes:

$$H(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \lambda^*(t_f)) = 0 \quad (2.31)$$

But the hamiltonian is constant along the optimal trajectory, therefore from (2.31) and (2.23) we obtain:

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t)) = 0, \quad t_0 \leq t \leq t_f \quad (2.32)$$

Minimum time problems. Bang-bang control

Example 3.1

Consider the problem of accelerating a skateboard in such way as to bring the skateboard at rest at given position in *minimum time*. Denote $x_1(t)$ the distance traveled at time t , $\dot{x}_1(t) = x_2(t)$ the velocity, and $\ddot{x}_1(t) = \dot{x}_2(t) = u(t)$ the acceleration. The model of the dynamical system is:

$$\dot{x}_1(t) = x_2(t); \quad \dot{x}_2(t) = u(t)$$

with the final states specified: $x_1(T) = x_2(T) = 0$. We want to determine the minimum final time T by minimizing:

$$J = \int_0^T 1 dt$$

Consider the constraints on control variable $-1 \leq u(t) \leq 1$.

The hamiltonian is:

$$H(\mathbf{x}, \mathbf{u}, \lambda) = 1 + \lambda_1(t)x_2(t) + \lambda_2(t)u(t)$$

By examining the form of the hamiltonian we can see that the term $\lambda_2(t)u(t)$, the only one which depends on $u(t)$, will decide the minimum value of H with respect to u .

The costate $\lambda_2(t)$ is a *switching function* because the optimal value of $u(t)$ changes when $\lambda_2(t)$ changes sign. The explanation follows below.

The costate equations are:

$$\begin{cases} \dot{\lambda}_1(t) = -\frac{\partial H}{\partial x_1} = 0 \\ \dot{\lambda}_2(t) = -\frac{\partial H}{\partial x_2} = -\lambda_1(t) \end{cases} \Rightarrow \begin{cases} \lambda_1(t) = C_1 \\ \lambda_2(t) = -C_1 t + C_2 \end{cases}$$

The switching function $\lambda_2(t)$ is a line and therefore it can change sign **at most once**. Then:

$$u^*(t) = \begin{cases} +1, & \lambda_2(t) < 0 \\ -1, & \lambda_2(t) > 0 \end{cases}$$

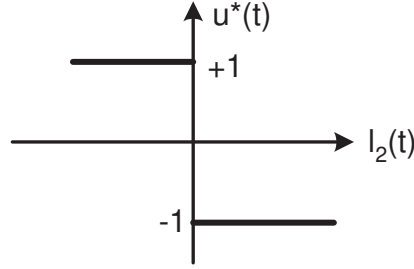


Figure 3.1: Bang-bang control

For $\lambda_2(t) < 0$, the term $\lambda_2(t)u(t)$ from the hamiltonian has its minimum value (negative) when $u(t)$ has its maximum positive value. Then for $u^*(t) = 1$ the hamiltonian is minimum for any $t \in [0, T]$ (we subtracted the maximum possible from the term $1 + \lambda_1(t)x_2(t)$ which is constant with respect to u). For $\lambda_2(t) > 0$, the term $\lambda_2(t)u(t)$ will be 'most negative' for $u^*(t) = -1$ and then the hamiltonian reaches the global minimum.

The optimal control is given now in terms of the costate vector $\lambda(t)$. We want the control to be given in terms of the state $\mathbf{x}(t)$ so we will have a closed-loop control system.

Solving the state equations:

$$u^*(t) = -1 \Rightarrow \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -1 \\ x_1(0) = x_{10}, x_2(0) = x_{20} \end{cases} \Rightarrow \begin{cases} x_1(t) = -\frac{t^2}{2} + x_{20}t + x_{10} \\ x_2(t) = -t + x_{20} \end{cases} \quad (3.1)$$

$$\Rightarrow x_1(t) = -\frac{1}{2}(x_{20} - t)^2 + x_{10} + \frac{x_{20}^2}{2} = -\frac{1}{2}(x_{20} - t)^2 + k_1 = -\frac{x_2^2(t)}{2} + k_1 \quad (3.2)$$

$$u^*(t) = +1 \Rightarrow \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = +1 \\ x_1(0) = x_{10}, x_2(0) = x_{20} \end{cases} \Rightarrow \begin{cases} x_1(t) = \frac{t^2}{2} + x_{20}t + x_{10} \\ x_2(t) = t + x_{20} \end{cases} \quad (3.3)$$

$$\Rightarrow x_1(t) = \frac{1}{2}(t + x_{20})^2 + x_{10} - \frac{x_{20}^2}{2} = \frac{1}{2}(x_{20} + t)^2 + k_2 = \frac{x_2^2(t)}{2} + k_2 \quad (3.4)$$

The equations (3.2) and (3.4) are a set of parabolas opening about the x_1 axis, as shown in Figure 3.2. Two parabolas pass through each point in $x_1 - x_2$

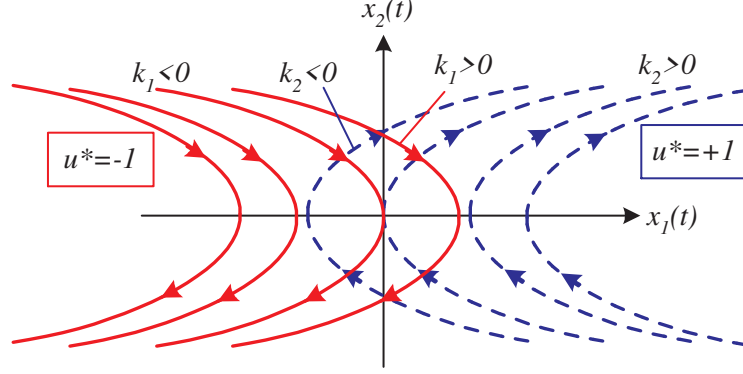


Figure 3.2: State space trajectories for bang-bang control

space, one for $u^* = -1$ and one for $u^* = +1$. For a given $\mathbf{x}(0) = [x_{10} \ x_{20}]^T$, using only $u^* = -1$ or $u^* = +1$ may not transfer the system from the initial state to the origin $\mathbf{x}(T) = \mathbf{0}$ (the target set).

The parabolas that pass through the origin are the ones for which $k_1 = k_2 = 0$, or, the ones for which the initial conditions satisfy:

$$x_{10} = -\frac{x_{20}^2}{2}, \text{ for } u^* = -1, \text{ or } x_{10} = \frac{x_{20}^2}{2}, \text{ for } u^* = +1 \quad (3.5)$$

The equation of the curve that transfer the system to the origin are obtained as follows:

$$u^*(t) = -1 \Rightarrow \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -1 \\ x_1(T) = x_2(T) = 0 \end{cases} \Rightarrow \begin{cases} x_1(t) = -\frac{(T-t)^2}{2} \\ x_2(t) = T-t \end{cases} \Rightarrow x_1(t) = -\frac{x_2^2(t)}{2}$$

$$u^*(t) = +1 \Rightarrow \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = +1 \\ x_1(T) = x_2(T) = 0 \end{cases} \Rightarrow \begin{cases} x_1(t) = \frac{(t-T)^2}{2} \\ x_2(t) = t-T \end{cases} \Rightarrow x_1(t) = \frac{x_2^2(t)}{2}$$

Since $t < T$, for $u^* = -1 \Rightarrow x_2(t) > 0$ and for $u^* = +1 \Rightarrow x_2(t) < 0$. Then the equation of the *switching locus* is written as:

$$x_1(t) = -\frac{1}{2}x_2(t)|x_2(t)| \quad (3.6)$$

and is presented in Figure 3.3.

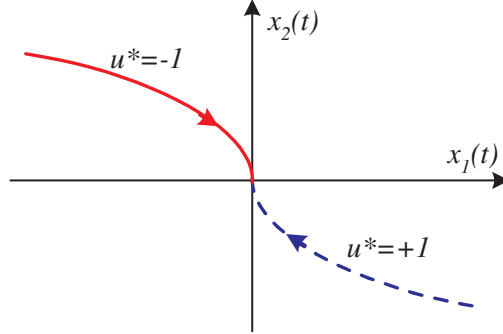


Figure 3.3: Switching locus

If, by chance, the initial state lies on the switching locus, then we have $u = \pm 1$ accordingly as we have $x_1 < > 0$. If, as in most cases, the initial state is not on the switching locus, then $u = \pm 1$ must be chosen to move the system toward the switching locus. By inspection, it is apparent that, above the switching locus we have $u = -1$ and below we have $u = +1$; a typical path from an initial state x_{10}, x_{20} is shown in the Figure 3.4.

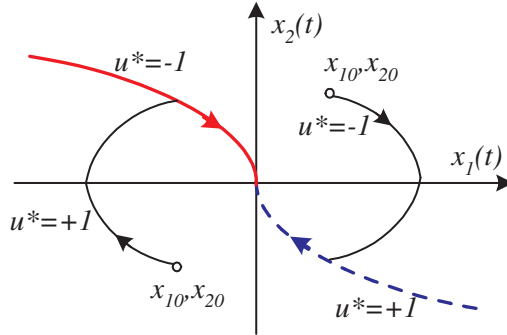


Figure 3.4: Examples of optimal trajectories

The optimal control law is applied according to the following rules:

- If $\mathbf{x}(0)$ is to the left of the switching curve, apply $u^* = +1$ until the curve is reached, then apply $u^* = -1$ until the origin is reached.
- If $\mathbf{x}(0)$ is to the right of the switching curve, apply $u^* = -1$ until the curve is reached, then apply $u^* = +1$ until the origin is reached.
- If $\mathbf{x}(0)$ is on the switching curve, apply $u = -\text{sign}(x_2(0))$ until the origin is reached.

- Apply $u = 0$ when the origin is reached.

We have obtained the optimal control law at any time t as a function of the state value. To express the optimal control law in a convenient form, let us define the *switching function*, obtained from equation (3.6):

$$s(\mathbf{x}(t)) = x_1(t) + \frac{1}{2}x_2(t)|x_2(t)| \quad (3.7)$$

Notice that:

- $s(\mathbf{x}(t)) > 0$ implies $\mathbf{x}(t)$ lies on the right of the switching locus
- $s(\mathbf{x}(t)) < 0$ implies $\mathbf{x}(t)$ lies on the left of the switching locus
- $s(\mathbf{x}(t)) = 0$ implies $\mathbf{x}(t)$ lies on the switching locus

In terms of this switching function, the optimal control law is:

- $u^* = -1$ for $\mathbf{x}(t)$ such that $s(\mathbf{x}(t)) > 0$
- $u^* = +1$ for $\mathbf{x}(t)$ such that $s(\mathbf{x}(t)) < 0$
- $u^* = -1$ for $\mathbf{x}(t)$ such that $s(\mathbf{x}(t)) = 0$ and $x_2(t) > 0$
- $u^* = +1$ for $\mathbf{x}(t)$ such that $s(\mathbf{x}(t)) = 0$ and $x_2(t) < 0$
- $u^* = 0$ for $\mathbf{x}(t) = 0$

An implementation of this optimal control law is shown in Figure 3.5.

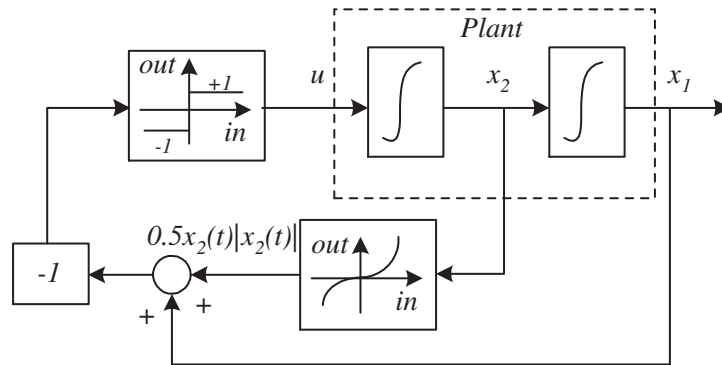


Figure 3.5: Implementation of the time-optimal control law

Minimum fuel problems

4.1 Minimum fuel problems

4.1.1 The statement of the problem

For a system described by the state equation:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad t_0 \leq t \leq t_f, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (4.1)$$

where $\mathbf{u}(t)$ is an $m \times 1$ control vector, a minimum fuel problem would require to determine the optimal control vector $\mathbf{u}^*(t)$ that minimizes the performance measure:

$$J = \int_{t_0}^{t_f} \left[\sum_{i=1}^m \alpha_i |u_i(t)| \right] dt \quad (4.2)$$

while satisfying the constraints:

$$-M \leq u_i(t) \leq M, \quad i = 1, 2, \dots, m \quad (4.3)$$

It is assumed that the fuel consumption is proportional to the magnitude of the control inputs.

4.1.2 Example of a minimum fuel problem

Example 4.1 *The plant described by the differential equation:*

$$\dot{x}(t) = -x(t) + u(t) \quad (4.4)$$

is to be transferred from an arbitrary initial state $x(0) = x_0$ to the origin, $x(T) = 0$, by a control that minimizes the performance measure:

$$J(u) = \int_0^T |u(t)| dt \quad (4.5)$$

The admissible controls are constrained by:

$$|u(t)| \leq 1 \quad (4.6)$$

and the final time T is fixed.

A more general case of this problem may be found in (Kirk, 2004). The solution of the problem is the optimal control signal that minimizes the hamiltonian, written as:

$$H(x(t), u(t), \lambda(t)) = |u(t)| + \lambda(t)(-x(t) + u(t)) \quad (4.7)$$

The terms in the hamiltonian that depend on the control $u(t)$ are given by (the reduced hamiltonian H_r):

$$H_r(u(t), \lambda(t)) = |u(t)| + \lambda(t)u(t) \quad (4.8)$$

or:

$$H_r = u(t) + \lambda(t)u(t), \text{ for } u(t) \geq 0 \quad (4.9)$$

$$H_r = -u(t) + \lambda(t)u(t), \text{ for } u(t) \leq 0 \quad (4.10)$$

- If $\lambda(t) > 1$, the minimum value of expression (4.9), which is a positive quantity, is 0 because $u(t) \geq 0$; but the minimum of (4.10) is a negative value and is attained for $u^*(t) = -1$.
- If $\lambda(t) = 1$, the expression (4.9) has a minimum at 0, with $u(t) = 0$; but (4.10) will be zero for any non-positive value of $u(t)$. Thus, any non-positive value of $u(t)$ will minimize the hamiltonian.
- If $0 < \lambda(t) < 1$, the minimum of (4.9) and (4.10) is zero (they are both positive) and is attained for $u^*(t) = 0$.
- If $\lambda(t) < -1$, the minimum value of expression (4.10), which is a positive quantity, is 0 because $u(t) \leq 0$; but the minimum of (4.9) is a negative value and is attained for $u^*(t) = +1$.

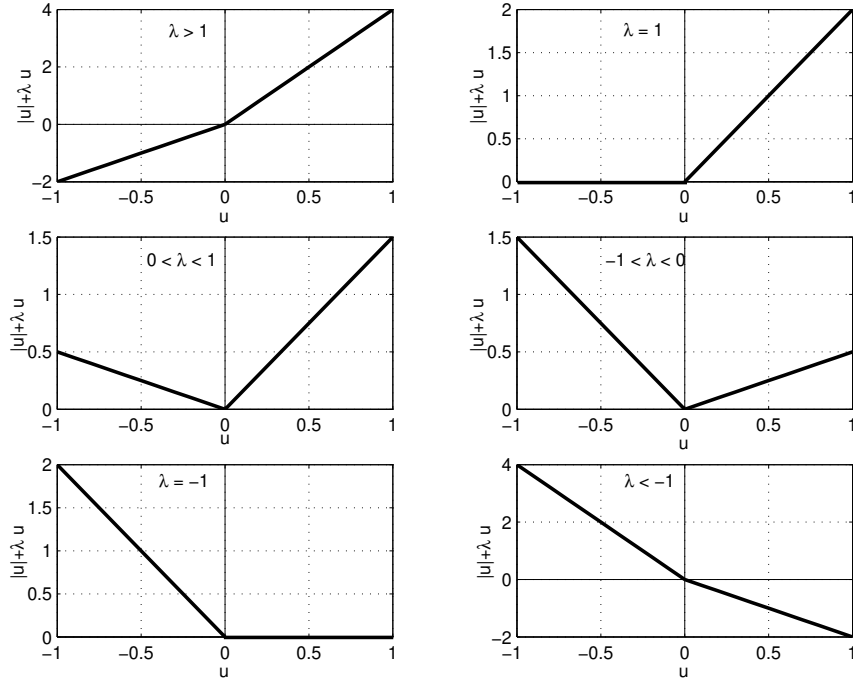


Figure 4.1: Reduced hamiltonian for various values of λ

- If $\lambda(t) = -1$, the expression (4.10) has a minimum at 0, with $u(t) = 0$; but (4.9) will be zero (minimum) for any non-negative value of $u(t)$.
- If $-1 < \lambda(t) < 0$, the minimum of (4.9) and (4.10) is zero (they are both positive) and is attained for $u^*(t) = 0$.

The above situations are depicted in Figure 4.1 and the conclusions follow directly from the plots.

Summarizing, the control signal that minimizes the hamiltonian (4.7) is:

$$u^*(t) = \begin{cases} +1, & \text{for } \lambda^*(t) < -1 \\ -1, & \text{for } \lambda^*(t) > 1 \\ 0, & \text{for } |\lambda^*(t)| < 1 \\ \text{undetermined non-positive value for } \lambda^*(t) = +1 \\ \text{undetermined non-negative value for } \lambda^*(t) = -1 \end{cases} \quad (4.11)$$

The costate equation and its solution are:

$$\dot{\lambda}(t) = -\frac{\partial H}{\partial x} = \lambda(t) \Rightarrow \lambda(t) = Ce^t \quad (4.12)$$

There are five possible trajectories for $\lambda^*(t)$ that affect the control signal $u^*(t)$, depending on the value of the constant of integration C , as shown in Figure 4.2, (Beale, 2001).

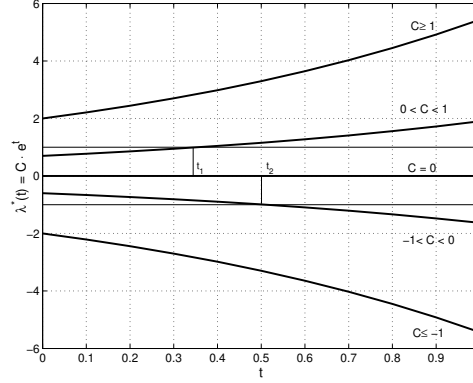


Figure 4.2: Forms of the costate

Because of the exponential form of the costate, the situation when $\lambda^*(t) = \pm 1$ cannot occur for a finite time interval. Therefore, the optimal control can be expressed as:

$$u^*(t) = \begin{cases} +1, & \text{for } \lambda^*(t) < -1 \\ -1, & \text{for } \lambda^*(t) > 1 \\ 0, & \text{for } |\lambda^*(t)| < 1 \end{cases} \quad (4.13)$$

The form *bang-off-bang* of the optimal control $u^*(t)$ as a function of the costate $\lambda^*(t)$ is shown in Figure 4.3.

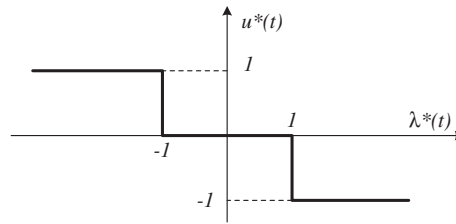


Figure 4.3: Bang-off-bang optimal control

Four of the λ^* -trajectories are shown in Figure 4.4 and the fifth is $\lambda^*(t) = 0$ for which $u^*(t) = 0$, $t \in [0, T]$.

The state trajectories can be found by substituting the admissible values of the control signal $u^*(t)$ into the state equation and solving for $x(t)$. This will determine whether or not the target set can be reached and if it depends on the initial conditions, (Beale, 2001).

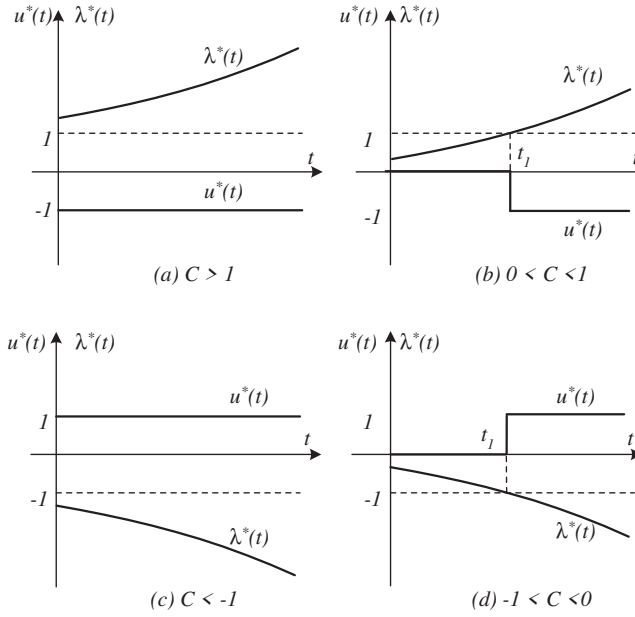


Figure 4.4: Forms of the costate and optimal control signals, (Kirk, 2004)

$u^*(t) = 0$. The solution of the state equation is obtained as follows:

$$\dot{x}(t) = -x(t), \quad x(0) = x_0, \Rightarrow x(t) = x_0 e^{-t} \quad (4.14)$$

The exponential form of the trajectory indicates that the system will approach the final state zero as time approaches infinity, but it will never attain the zero value in a finite time. Therefore, when no control signal is applied, or $u^*(t) = 0$, this cannot be the only value used.

$u^*(t) = +1$. The solution of the state equation is :

$$\dot{x}(t) = -x(t) + 1, \quad x(0) = x_0, \Rightarrow x(t) = 1 + (x_0 - 1)e^{-t} \quad (4.15)$$

This state trajectory moves towards 1 as time increases and it will pass through the value 0 only if $x_0 < 0$.

For a fixed final time T , the state will get to the origin only for a particular value of the initial condition, determined from:

$$x(T) = 0 = 1 + (x_0 - 1)e^{-T} \Rightarrow x_0 = 1 - e^T \quad (4.16)$$

For the rest of the initial conditions switching is needed.

$u^*(t) = -1$. The solution of the state equation is :

$$\dot{x}(t) = -x(t) - 1, x(0) = x_0, \Rightarrow x(t) = -1 + (x_0 + 1)e^{-t} \quad (4.17)$$

This state trajectory moves towards -1 as time increases. It will pass through the value 0 only if $x_0 > 0$.

The state will attain the origin using only the control value $u^*(t) = -1$, in a fixed time T , only for the initial state obtained as:

$$x(T) = 0 = -1 + (x_0 + 1)e^{-T} \Rightarrow x_0 = -1 + e^T \quad (4.18)$$

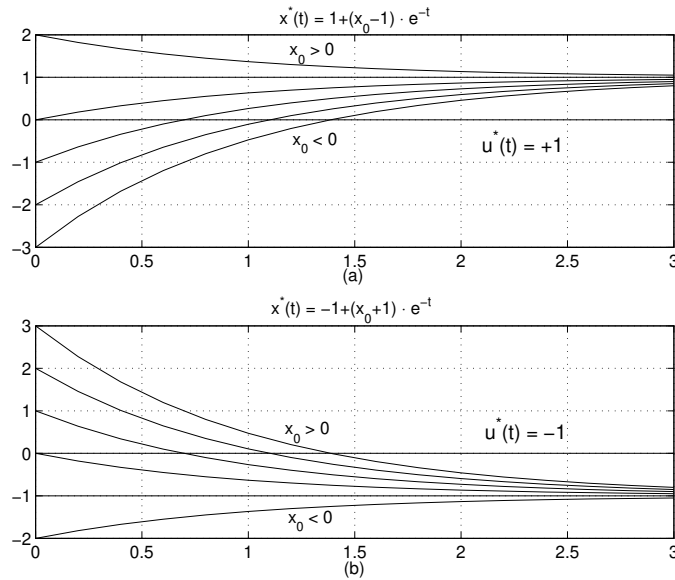


Figure 4.5: State trajectories for different initial conditions. (a) $u^*(t) = +1$
(b) $u^*(t) = -1$

Figure 4.5 (a) and (b) show the state trajectories for $u^*(t) = -1$ and $u^*(t) = +1$. They indicate that the target set $x(T) = 0$ can be reached in most of the cases (when the initial condition does not have one of the values given by (4.16) or (4.18)) only after switching the control value from $u^*(t) = 0$ to $u^*(t) = -1$ (if the initial state $x_0 > 0$) or to $u^*(t) = +1$ (if the initial state $x_0 < 0$).

4.1.3 Determining the switching time

Initial condition x_0 is positive: $x_0 > 0$. The control $u^*(t) = 0$ will not drive the system to the origin $x(T) = 0$ in a finite time. As shown in Figure 4.4b), either $u^*(t) = -1$ or $u^*(t) = [0, -1]$ must be used.

Let $t = t_1$ be the switching time from $u^*(t) = 0$ to $u^*(t) = -1$. The state value at time t_1 is obtained from (4.14) for the final time $t = t_1$.

$$x(t_1) = x_0 e^{-t_1} \quad (4.19)$$

After switching, the state equation (4.17) will be solved for the time interval $[t_1, T]$ with the initial condition $x(t_1)$:

$$\dot{x}(t) = -x(t) - 1, \quad x(t_1) = x_0 e^{-t_1} \quad (4.20)$$

and the solution is:

$$x(t) = -1 + (x_0 + e^{t_1})e^{-t} \quad (4.21)$$

The switching time t_1 is obtained by substituting $t = T$ into (4.21) and letting it equal to zero (the target set), as shown below:

$$x(T) = -1 + (x_0 + e^{t_1})e^{-T} = 0 \quad (4.22)$$

$$(x_0 + e^{t_1})e^{-T} = 1, \Rightarrow x_0 + e^{t_1} = e^T, \Rightarrow t_1 = \ln(e^T - x_0) \quad (4.23)$$

Initial condition x_0 is negative: $x_0 < 0$. The switching time t_1 can be obtained for negative initial conditions in the same manner.

In this case (see Figure 4.4d)), the optimal control is zero $u^*(t) = 0$ for $t \in [0, t_1]$ and $u^*(t) = +1$ for $t \in [t_1, T]$.

The switching time results from the calculations given below:

$$\dot{x}(t) = -x(t) + 1, \quad x(t_1) = x_0 e^{-t_1} \quad (4.24)$$

$$x(t) = 1 + (x_0 - e^{t_1})e^{-t} \quad (4.25)$$

$$x(T) = 1 + (x_0 - e^{t_1})e^{-T} = 0 \quad (4.26)$$

$$x_0 - e^{t_1} = -e^T, \Rightarrow t_1 = \ln(e^T + x_0) \quad (4.27)$$

4.1.4 Open-loop control

From all the calculations above, the optimal control is:

$$u^*(t) = \begin{cases} 0, & \text{for } x_0 > 0 \text{ and } t < \ln(e^T - x_0) \\ -1, & \text{for } x_0 > 0 \text{ and } \ln(e^T - x_0) < t < T \\ 0, & \text{for } x_0 < 0 \text{ and } t < \ln(e^T + x_0) \\ +1, & \text{for } x_0 < 0 \text{ and } \ln(e^T + x_0) < t < T \end{cases} \quad (4.28)$$

The optimal control (4.28) is an open-loop form since the current value of the state $x(t)$ is not used to determine the control signal.

Closed-loop control would be preferable to reduce the effects of disturbances.

4.1.5 Closed-loop control

The closed-loop control law will be obtained by solving the state equation for $u^*(t) = -1$ and then for $u^*(t) = +1$, with the final condition $x(T) = 0$. This is based on the observation that during the last part of the time interval $[t_1, T]$, the optimal control is either $+1$ or -1 depending on whether the initial state is negative or positive.

For $x(t) > 0$:

$$\dot{x}(t) = -x(t) - 1, \quad x(T) = 0, \Rightarrow x(t) = -1 + e^{T-t}, \quad t > t_1 \quad (4.29)$$

For $x(t) < 0$:

$$\dot{x}(t) = -x(t) + 1, \quad x(T) = 0, \Rightarrow x(t) = 1 - e^{T-t}, \quad t > t_1 \quad (4.30)$$

During the first part of the interval $t \in [0, t_1]$, the optimal control is zero and

$$x(t) = x_0 e^{-t}, \quad t < t_1 \quad (4.31)$$

The switching of the control from 0 to -1 occurs when the curve (4.31) intersect curve (4.29). We denote $x(t)$ from (4.29) with:

$$s(t) = -1 + e^{T-t} \quad (4.32)$$

and call it the *equation of the switching curve*. The control will be 0 for all state values smaller than $s(t)$ and it will switch to -1 when they become equal.

For negative values of the states $x(t) < 0$ we apply a similar reasoning and find out that the control is 0 for $0 < x(t) < s(t)$ and it switches to +1 for $x(t) \leq s(t)$.

The closed-loop form of the optimal control can be written as:

$$u^*(t) = \begin{cases} -1, & \text{for } x(t) \geq s(t) \\ 0, & \text{for } |x(t)| < s(t) \\ +1, & \text{for } x(t) \leq -s(t) \end{cases} \quad (4.33)$$

Figure 4.6 shows the solution for $x_0 > 0$.

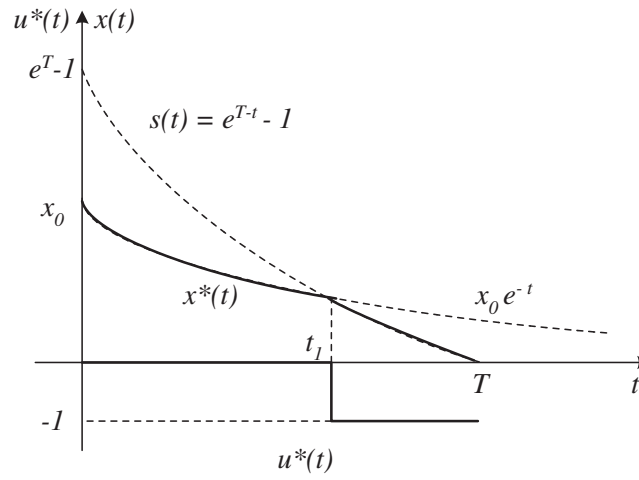


Figure 4.6: The optimal trajectory and the optimal control

Exercises

1. (Owens, 1981) Verify that for the scalar plant

$$\dot{x}(t) = -x(t) + u(t), \quad x(0) = 1, \quad |u(t)| \leq M \quad t \in [0, 1] \quad (5.1)$$

the admissible controller $u^*(t)$ that minimizes the performance measure:

$$J = x(1) + \int_0^1 u^2(t) dt \quad (5.2)$$

takes the form (open-loop):

- (a) if $2M \geq 1$:

$$u^*(t) = -\frac{1}{2}e^{t-1} \quad (5.3)$$

- (b) if $2M < 1$:

$$u^*(t) = \begin{cases} -\frac{1}{2}e^{t-1}, & \text{for } 0 \leq t \leq 1 + \ln 2M \\ -M, & \text{for } 1 + \ln 2M \leq t \leq 1 \end{cases} \quad (5.4)$$

2. (Kirk, 2004) A differential equation that describes the leaky reservoir shown in Figure 5.1 is:

$$\dot{x}(t) = -0.1x(t) + u(t) \quad (5.5)$$

where $x(t)$ is the height of the water and $u(t)$ is the net inflow rate of water at time t . Assume $0 \leq u(t) \leq M$.

Find the optimal control law if it is desired to minimize:

$$J = \int_0^{100} -x(t) dt \quad (5.6)$$

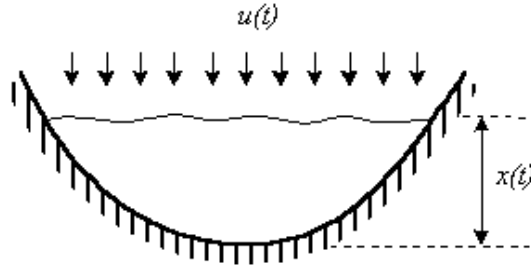


Figure 5.1: Reservoir

3. (Owens, 1981) Consider the plant with the state equations:

$$\dot{x}_1(t) = x_2(t) \quad (5.7)$$

$$\dot{x}_2(t) = u(t) \quad (5.8)$$

Find the optimal control law that brings the system from the initial state $\mathbf{x}(0) = [1, 0]^T$ to the final state $\mathbf{x}(1) = [0, 0]^T$, and minimizes the performance measure:

$$J = \frac{1}{2} \int_0^1 u^2(t) dt \quad (5.9)$$

The control is constrained by: $-M \leq u(t) \leq M$.

4. Minimize the performance measure:

$$J = \int_0^1 x(t) dt \quad (5.10)$$

subject to the state equation:

$$\dot{x}(t) = u(t), \quad x(0) = 1 \quad (5.11)$$

and the control constraint: $-1 \leq u(t) \leq 1$.

5. Minimize the performance measure:

$$J = \int_0^2 (u^2(t) + 3u(t) - 2x(t)) dt \quad (5.12)$$

subject to the state equation:

$$\dot{x}(t) = x(t) + u(t), \quad x(0) = 5 \quad (5.13)$$

and the control constraint: $0 \leq u(t) \leq 2$.

6. (Owens, 1981) Verify that the solution of the problem:

$$\dot{x}(t) = -x(t) + u(t), \quad x(0) = 0 \quad (5.14)$$

$$J = 2x(1) + \int_0^1 \left(x(t) + \frac{1}{2}u^2(t) \right) dt \quad (5.15)$$

in the presence of the constraint $|u(t)| \leq M$ is the optimal control:

$$u^*(t) = \begin{cases} -\lambda(t) & \text{if } |\lambda(t)| \leq 1 \\ -M \text{sign}(\lambda(t)) & \text{if } |\lambda(t)| > 1 \end{cases} \quad (5.16)$$

where $\lambda(t) = 1 + e^{t-1}$ is the solution of the costate equation

$$\dot{\lambda}(t) = \lambda(t) - 1, \quad \lambda(0) = 2 \quad (5.17)$$

Hence verify that:

a)

$$u^*(t) = -M \quad \text{if} \quad M \leq 1 + e^{-1} \quad (5.18)$$

b)

$$u^*(t) = -1 - e^{t-1} \quad \text{if} \quad M \geq 2 \quad (5.19)$$

c)

$$u^*(t) = \begin{cases} -1 - e^{t-1} & \text{if } 0 \leq t \leq \ln(M-1) \\ -M & \text{if } t > 1 + \ln(M-1) > 1 \end{cases} \quad (5.20)$$

if $1 + e^{-1} \leq M \leq 2$.

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